Logarithm Contest Questions

March 12, 2019 8:33 AM

1. Determine all values of "x" such that: $\log_{2x}\left(48\sqrt[3]{3}\right) = \log_{3x}\left(162\sqrt[3]{2}\right)$ Euclid

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$\log_{2x} 16 \times 3^{\frac{4}{3}} = \log_{3x}\left(3^4 \times 2^{\frac{4}{3}}\right)$

$= \dfrac{\log\left(2^4 \times 3^{\frac{4}{3}}\right)}{\log 2x} = \dfrac{\log 3^4 + \log 2^{\frac{4}{3}}}{\log 3x}$

$= \dfrac{4\log 2 + \frac{4}{3}\log 3}{\log 2 + \log x} = \dfrac{4\log 3 + \frac{4}{3}\log 2}{\log 3 + \log x}$

$\dfrac{3\log 2 + \log 3}{\log 2 + \log x} = \dfrac{3\log 3 + \log 2}{\log 3 + \log x}$

$\dfrac{3A + B}{A + C} = \dfrac{3B + A}{B + C}$

$\boxed{\begin{matrix} A = \log 2 \\ B = \log 3 \\ C = \log x \end{matrix}}$

MAKE A SUBSTITUTION TO SIMPLIFY THE ALGEBRA!

$(3A+B)(B+C) = (3B+A)(A+C)$

$\cancel{3AB} + B^2 + \cancel{3AC} + \cancel{BC} = \cancel{3AB} + A^2 + \cancel{3BC} + \cancel{AC}$

$B^2 + 2AC - 2BC - A^2 = 0$

$2C(A - B) = A^2 - B^2$

$C = \dfrac{(A+B)\cancel{(A-B)}}{2\cancel{(A-B)}}$

$\log x = \dfrac{\log 2 + \log 3}{2}$

$\log x^2 = \log 6$

$x^2 = 6$

$\boxed{x = \sqrt{6}}$

① EARLIER MISTAKES

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$\dfrac{3\log 2 + \log 3}{\log 2x} = \dfrac{\log 3^3 + \log 2}{\log 3x}$

$\dfrac{3\log(2/3)}{} = \log\left(\frac{2}{3}\right)$

$\dfrac{3\log 2 - 3\log 3}{\log 2x} \quad \dfrac{\log 2x}{\log 3x} = \dfrac{\log 2}{\log 3x} - \dfrac{\log 3}{\log 2x}$

No! b/c they have diff. BASES!

① $\log_{2x}\left(48\sqrt[3]{3}\right) = \log_{2x}\left(16 \times 3 \times \sqrt[3]{3}\right)$

$= \log_{2x} 2^4 \times 3^{1+\frac{1}{3}}$

$= \log_{2x} 2^4 \times 3^{\frac{4}{3}}$

$= 4\log_{2x} 2 + \frac{4}{3}\log_{2x} 3$

② $\log_{3x}\left(81 \times 2^1 \times \sqrt[3]{2}\right)$

$\log_{3x}\left(3^4 \times 2^{\frac{4}{3}}\right)$

$= \log_{3x} 3^4 + \log_{3x} 2^{\frac{4}{3}}$

$= 4\log_{3x} 3 + \frac{4}{3}\log_{3x} 2$

$4\log_{2x} 2 + \frac{4}{3}\log_{2x} 3 = 4\log_{3x} 3 + \frac{4}{3}\log_{3x} 2$

$3\log_{2x} 2 + \log_{2x} 3 = 3\log_{3x} 3 + \log_{3x} 2$

$\log_{2x}(24) = \log_{3x}(54)$

$\dfrac{\log 2x}{\log 24} = \dfrac{\log 3x}{\log 54}$

$\log_{24} 2 + \log_{24} x = \log_{54} 3 + \log_{54} x$

$\dfrac{\log x}{\log 24} - \dfrac{\log x}{\log 54} = \dfrac{\log 3}{\log 54} - \dfrac{\log 2}{\log 24}$

$\log x\left(\dfrac{1}{\log 24} - \dfrac{1}{\log 54}\right) = \left(\dfrac{\log 3}{\log 54} - \dfrac{\log 2}{\log 24}\right)$

$3\log_{2x} 2 - 3\log_{3x} 3 = \log_{3x} 2 - \log_{2x} 3$

$\cancel{3\left[\log_{2x}\frac{2}{3}\right] = \log_{3x}\frac{2}{3}}$ NO!

DIFF. BASES! CAN'T COMBINE THEM!

$3\left[\dfrac{\cancel{\log\frac{2}{3}}}{\log 2x}\right] = \left[\dfrac{\cancel{\log\frac{2}{3}}}{\log 3x}\right]$

$3\log 3x = \log 2x$

$\log(3x)^3 = \log 2x$

$(3x)^3 = 2x$

$27x^3 - 2x = 0$

$x(27x^2 - 2) = 0$

$x = 0 \quad x^2 = \frac{2}{27}$

$x = \sqrt{\frac{2}{27}}$

$\dfrac{3}{\log 2 + \log x} = \dfrac{1}{\log 3 + \log x}$

$3\log 3 + 3\log x = \log 2 + \log x$

$2\log x = \log 2 - \log 27$

$2\log x = \log\left(\frac{2}{27}\right)$

$x^2 = \frac{2}{27}$

$\boxed{x = \sqrt{\frac{2}{27}}} \quad \dfrac{\sqrt{2}}{3\sqrt{3}} = \dfrac{\sqrt{6}}{3 \times 3} = \dfrac{\sqrt{6}}{9}$

2. Determine all real numbers X>0 for which $\log_4 x - \log_x 16 = \frac{7}{6} - \log_x 8$ (Euclid)

$\dfrac{\log x}{2\log 2} - \dfrac{4\log 2}{\log x} = \dfrac{7}{6} - \dfrac{3\log 2}{\log x}$

$\dfrac{1}{2}\dfrac{\log x}{\log 2} = \dfrac{7}{6} + 4\dfrac{\log 2}{\log x} - \dfrac{3\log 2}{\log x}$

$\dfrac{1}{2}\dfrac{\log x}{\log 2} = \dfrac{7}{6} + \dfrac{\log 2}{\log x}$

SUBSTITUTE $A = \dfrac{\log x}{\log 2} \quad \dfrac{1}{A} = \dfrac{\log 2}{\log x}$

$\frac{1}{2}A = \frac{7}{6} + \frac{1}{A}$

$3A^2 = 7A + 6$

$3A^2 - 7A - 6 = 0$

$3 \;—\; -3$
$1 \;—\; 2$

$(3A+2)(A-3) = 0$

$A = -\frac{2}{3} \quad A = 3$

$\log_2 x = -\frac{2}{3} \quad \log_2 x = 3$

$2^{-\frac{2}{3}} = x \quad 2^3 = x$

$\boxed{\dfrac{1}{\sqrt[3]{4}} = x} \quad \boxed{8 = x}$

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3. Determine all real numbers "x" for which $(\log_{10} x)^{\log_{10}(\log_{10} x)} = 10{,}000$ (Euclid)

$(\log_{10} x)^{\log_{10}(\log_{10} x)} = 10{,}000 \rightarrow \log_{\log_{10} x} 10{,}000 = \log_{10}(\log_{10} x)$

$\boxed{A^B = C \rightarrow \log_A C = B}$

$\dfrac{\log 10{,}000}{\log(\log_{10} x)} = \dfrac{\log(\log_{10} x)}{\log 10}$

$\log 10{,}000 \times \log 10 = [\log(\log_{10} x)]^2$

$\sqrt{4} = \log(\log_{10} x)$

$\pm 2 = \log(\log_{10} x)$

$10^2 = \log_{10} x \quad 10^{-2} = \log_{10} x$

$\boxed{10^{100} = x} \quad \boxed{10^{.01} = x}$

* WHEN SOLVING LOG EQN'S, MAKE SURE YOU STICK TO THE LOG RULES! NO SHORTCUTS.

• ALSO USED BASIC ALGEBRA WITH FACTORING!

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4. The solution of the equation $7^{x+7} = 8^x$ can be expressed in the form $x = \log_b 7^7$, what is the value of "b"?(AMC12)

① $7^{x+7} = 8^x$

CHANGE TO LOG FORM!

$7^x \cdot 7^7 = 8^x$

$7^7 = \dfrac{8^x}{7^x}$

$7^7 = \left(\frac{8}{7}\right)^x$

$\log_{\left(\frac{8}{7}\right)} 7^7 = x$

$b = 8/7$

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5. What is the value of "x" for which the equation is true? $\dfrac{1}{\log_2 a} + \dfrac{1}{\log_3 a} + \dfrac{1}{\log_4 a} = 1$ (AMC12)

$\dfrac{1}{\left(\frac{\log a}{\log 2}\right)} + \dfrac{1}{\left(\frac{\log a}{\log 3}\right)} + \dfrac{1}{\left(\frac{\log a}{\log 4}\right)} = 1$

$\dfrac{\log 2}{\log a} + \dfrac{\log 3}{\log a} + \dfrac{\log 4}{\log a} = 1$

$\dfrac{\log(2 \times 3 \times 4)}{\log a} = 1$

$\log(24) = \log a$

$\boxed{24 = a}$

6. The sequence of terms forms an arithmetic progression. What is the value of "x"?
$\log_{12} 162,\ \log_{12} x,\ \log_{12} y,\ \log_{12} z,\ \log_{12} 1250$

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① $\frac{\log 162}{\log 12}, \frac{\log x}{\log 12}, \frac{\log y}{\log 12}, \frac{\log z}{\log 12}, \frac{\log 1250}{\log 12}.$

+d +d +d +d

② $\frac{\log 162}{\log 12} + 4d = \frac{\log 1250}{\log 12}.$

$4d = \frac{\log(1250/162)}{\log 12}.$

$\cancel{4}d = \frac{\cancel{4}\log(5/3)}{\log 12}.$

$\boxed{d = \frac{\log(5/3)}{\log 12}}$

1250 = 125 × 10 = $5^3 \times 5 \times 2$

162 = 81 × 2 = $3^4 \times 2$.

$\frac{1250}{162} = \frac{5^4}{3^4}$

$\therefore \log 1250/162 = \log(5/3)^4 = 4\log(5/3)$

③ $\log_{12} 162 + d = \log_{12} x$

$\frac{\log 162}{\log 12} + \frac{\log(5/3)}{\log 12} = \frac{\log x}{\log 12}.$

$\frac{\log\left(2 \times \cancel{3^4}^{3^3} \times \frac{5}{\cancel{3}}\right)}{\cancel{\log 12}} = \frac{\log x}{\cancel{\log 12}}$

$2 \times 3^3 \times 5 = x$

$\boxed{270 = x}$

7. Let "x", "y" and "z" all exceed 1, and let "w" be a positive number such that $\log_x w = 24$, $\log_y w = 40$, and $\log_{xyz} w = 12$. Find $\log_z w$.

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① $\log_{xyz} w = 12$

$\frac{\log w}{\log xyz} = 12.$

$\frac{\log xyz}{\log w} = \frac{1}{12}.$

$\log_w x + \log_w y + \log_w z = \frac{1}{12}.$

② $\log_x w = 24$

$\frac{\log x}{\log w} = \frac{1}{24}$

③ $\log_y w = 40$

$\frac{\log y}{\log w} = \frac{1}{40}$

④ SUBSTITUTE INTO 1st FORMULA

$\frac{1}{24} + \frac{1}{40} + (k) = \frac{1}{12}$ where $k = \log_w z$

$k = \frac{1}{12} - \frac{1}{24} - \frac{1}{40}$

$k = \frac{1}{24} - \frac{1}{40} = \frac{1}{8 \times 3} - \frac{1}{8 \times 5}$

$k = \frac{5}{8 \times 3 \times 5} - \frac{3}{8 \times 3 \times 5}$

$k = \frac{\cancel{2}^{1}}{\cancel{8}_{4} \times 3 \times 5} = \frac{1}{60}$

$\log_w z = \frac{1}{60}$

$\boxed{\log_z w = 60}$

8. Determine all pairs (a,b) of real numbers that satisfy the following system of equations. Give your answers as pairs of simplified exact numbers. (Euclid)
$\sqrt{a} + \sqrt{b} = 8$
$\log_{10} a + \log_{10} b = 2$

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① $\sqrt{a} + \sqrt{b} = 8$

② $\log(a \times b) = 2$

$\boxed{a \times b = 100}$

$\boxed{a = \frac{100}{b}}$

SUB ③ $\sqrt{\frac{100}{b}} + \sqrt{b} = 8.$

$\frac{10}{\sqrt{b}} + \sqrt{b} = 8.$

let $k = \sqrt{b}$ $\frac{10}{k} + k = 8$

$10 + k^2 = 8k$

$k^2 - 8k + 10 = 0$

$k = \frac{8 \pm \sqrt{64 - 40}}{2}$

$k = \frac{8 \pm 2\sqrt{6}}{2}$

$\boxed{k = 4 \pm \sqrt{6}}$

$\sqrt{b} = 4 + \sqrt{6}$ or $\sqrt{b} = 4 - \sqrt{6}.$

$b = (4+\sqrt{6})(4+\sqrt{6})$

$= 16 + 8\sqrt{6} + 6$

$\boxed{b = 22 + 8\sqrt{6}}$

$b = (4-\sqrt{6})(4-\sqrt{6})$

$b = 16 - 8\sqrt{6} + 6$

$\boxed{b = 22 - 8\sqrt{6}}$

$a = \frac{100}{22 + 8\sqrt{6}}$

$a = \frac{50(11 - 4\sqrt{6})}{(11 + 4\sqrt{6})(11 - 4\sqrt{6})}$

$a = \frac{50(11 - 4\sqrt{6})}{121 - 96}$

$a = \frac{50(11 - 4\sqrt{6})}{25}$

$a = 22 - 8\sqrt{6}$

16
6
60
36

11
121
96
25

$\therefore (a, b) = (22 - 8\sqrt{6}, 22 + 8\sqrt{6})$

$= (22 + 8\sqrt{6}, 22 - 8\sqrt{6})$

9. Consider the following system of equations in which all logarithms have base 10:
   a) If a=-4, b=4, and c=-18, solve the system of equations
   b) Determine all triples (a,b,c) of real numbers for which the system of equations has infinite number of solutions (x,y,z) (Euclid)

$$(\log x)(\log y) - 3\log 5y - \log 8x = a$$
$$(\log y)(\log z) - 4\log 5y - \log 16z = b$$
$$(\log z)(\log x) - 4\log 8x - 3\log 625z = c$$

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a) $a=-4 \quad b=4 \quad c=-18$.

$(\log x = A \quad \log y = B \quad \log z = C)$ MAKE SUBSTITUTIONS!

$\boxed{\log 5 + \log 2 = 1}$

① $A \times B - 3\log 5 - 3B - \log 8 - A = -4. \longrightarrow AB - 3B - A - 3(\log 5 + \log 2) = -4 \rightarrow AB - 3B - A - 3 = -4$

② $B \times C - 4\log 5 - 4B - \log 16 - C = 4 \longrightarrow BC - 4B - C - 4(\log 5 + \log 2) = 4. \rightarrow BC - 4B - C - 4 = 4$

③ $A \times C - 4\log 8 - 4A - 3\log 5^4 - 3C = -18 \longrightarrow AC - 4A - 3C - 12(\log 5 + \log 2) = -18 \rightarrow AC - 4A - 3C - 12 = -18$

① $AB - A - 3B + 3 - 6 = -4$

$A(B-1) - 3(B-1) = 2.$

$\boxed{(A-3)(B-1) = 2}$

② $BC - 4B - C + 4 - 8 = 4$

$B(C-4) - (C-4) = 12.$

$\boxed{(B-1)(C-4) = 12}$

③ $AC - 4A - 3C + 12 - 24 = -18$

$A(C-4) - 3(C-4) = 6.$

$\boxed{(A-3)(C-4) = 6}$

i) $\dfrac{\cancel{(B-1)}(C-4)}{\cancel{(B-1)}(A-3)} = 6.$

ii) $\dfrac{C-4}{\cancel{A-3}}\left(\cancel{(A-3)}(C-4)\right) = 36$

$(C-4)^2 = 36$

$\boxed{C-4 = \pm 6}$

$C = 4 \pm 6$

$C = 10$ or -2.

$\boxed{\log z = 10 \text{ or } \log z = -2}$

III) $\dfrac{(B-1)\cancel{(C-4)}}{(A-3)\cancel{(C-4)}} = 2$

$\left(\dfrac{B-1}{A-3}\right)(A-3)(B-1) = 4$

$(B-1)^2 = 4$

$B-1 = \pm 2$

$B = 1 \pm 2$

$\log y = 3$ or $\log y = -1$

$\boxed{y = 10^3 \text{ or } y = 10^{-1}}$

IV) $(A-3) = 1$ or -1.

$A = 3 \pm 1$

$A = 4$ or 2

$\log x = 4$ or $\log x = 2$

$\boxed{x = 10^4 \qquad x = 10^2.}$

$\boxed{\therefore (x, y, z) = (10^4, 10^3, 10^{10}) \text{ or } (10^2, 10^{-1}, 10^{-2})}$