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Section 3.7 Trigonometric Proofs

$\sin^2\theta+\cos^2\theta=1$ $1+\cot^2\theta=\csc^2\theta$ $\tan^2\theta+1=\sec^2\theta$ $\cos 2\theta=\cos^2\theta-\sin^2\theta$

$\sin(a-b)=\sin a\cos b-\sin b\cos a$ $\cos(a+b)=\cos a\cos b-\sin b\sin a$ $\cos 2\theta=1-2\sin^2\theta$

$\sin(a+b)=\sin a\cos b+\sin b\cos a$ $\cos(a-b)=\cos a\cos b+\sin b\sin a$ $\cos 2\theta=2\cos^2\theta-1$

$\sin 2\theta=2\sin\theta\cos\theta$

1. Write each of the following as a single trigonometric function:

a) $2\sin\frac{\pi}{3}\cos\frac{\pi}{3} = \sin\left(\frac{2\pi}{3}\right)$	b) $\sin^2\frac{\pi}{2}-\cos^2\frac{\pi}{2} = -\cos\pi$	c) $1-2\sin^2\frac{\pi}{3} = \cos\frac{2\pi}{3}$
d) $1+\cot^2\frac{\pi}{3} = \frac{1}{\sin^2\left(\frac{\pi}{3}\right)}$	e) $\cos^2\pi-\sin^2\pi = \cos 2\pi$	f) $\sin\frac{\pi}{4}\cos\frac{\pi}{4} = \frac{1}{2}\left(\sin\frac{\pi}{2}\right)$

2. Simplify the following trigonometric expressions in terms of sine and cosine

a) $\cot^2 x\sin^2 x+\cos^2 x$ $\frac{\cos^2 x}{\cancel{\sin^2 x}}\cdot\cancel{\sin^2 x}+\cos^2 x$ $= 2\cos^2 x$.	b) $\cot x+\tan x$ $\frac{\cos x}{\sin x}+\frac{\sin x}{\cos x}$ $= \frac{\cos^2 x+\sin^2 x}{\cos x\sin x} = \frac{1}{\cos x\cdot\sin x}$
c) $\left(\frac{\sec x}{\sin x}-\cot x\right)(\sin x-\csc x)$ $\left(\frac{1}{\cos x\sin x}-\frac{\cos x}{\sin x}\right)\left(\sin x-\frac{1}{\sin x}\right)$ $\left(\frac{1-\cos^2 x}{\cos x\sin x}\right)\left(\frac{\sin^2 x-1}{\sin x}\right)$ $\left(\frac{\cancel{\sin^2 x}}{\cancel{\cos x}\cancel{\sin x}}\right)\left(\frac{\cancel{\cos^2 x}}{\cancel{\sin x}}\right) = \cos x$.	d) $\frac{\sec x-\cos x}{\csc x-\sin x}$ $\frac{\frac{1}{\cos x}-\cos x}{\frac{1}{\sin x}-\sin x}$ $= \frac{\left(\frac{1-\cos^2 x}{\cos x}\right)}{\left(\frac{1-\sin^2 x}{\sin x}\right)}$ $= \left(\frac{\sin^2 x}{\cos x}\right)\div\left(\frac{\cos^2 x}{\sin x}\right)$ $= \frac{\sin^2 x}{\cos x}\times\frac{\sin x}{\cos^2 x}$ $= \frac{\sin^3 x}{\cos^3 x}$ //
e) $\frac{\sec x}{1-\cos x} = \frac{\left(\frac{1}{\cos x}\right)}{1-\cos x}$ $= \frac{1}{\cos x(1-\cos x)}$	f) $\frac{\csc^2 x+\sec^2 x}{\csc x\cdot\sec x}$ $\frac{\frac{1}{\sin^2 x}+\frac{1}{\cos^2 x}}{\left(\frac{1}{\sin x}\cdot\frac{1}{\cos x}\right)}$ $= \frac{\cos^2 x+\sin^2 x}{\sin^2 x\cdot\cos^2 x}\div\left(\frac{1}{\sin x}\cdot\frac{1}{\cos x}\right)$ $\frac{1}{\cancel{\sin^2 x}\cdot\cancel{\cos^2 x}}\times(\cancel{\sin x}\cdot\cancel{\cos x})$ $= \frac{1}{\sin x\cdot\cos x}$.
g) $\cos(A+B)\cdot\cos B+\sin(A+B)\cdot\sin B$ let $m=A+B$ $n=B$. $= \cos(m)\cos(n)+\sin(m)\sin n = \cos(m-n)$ $= \cos(A+B-B)$ $= \cos A$ //	h) $\csc x\cdot\cot x\cdot\sec x\cdot\sin x$ $\left(\frac{1}{\sin x}\right)\left(\frac{\cancel{\cos x}}{\cancel{\sin x}}\right)\left(\frac{1}{\cancel{\cos x}}\right)\cancel{\sin x}$ $= \frac{1}{\sin x}$
i) $\frac{\sin 3x}{\sin x}-\frac{\cos 3x}{\cos x}$ let $A=x$, $B=3x$ $\frac{\cos x\sin 3x-\sin x\cos 3x}{\sin x\cos x} = \frac{\cos A\sin B-\sin A\cos B}{\sin A\cos A}$ $= \frac{\sin(A-B)}{\sin A\cos A} = \frac{\sin(3x-x)}{\sin x\cos x} = \frac{\sin 2x}{\sin x\cos x}$ $= \frac{2\cancel{\sin x}\cancel{\cos x}}{\cancel{\sin x}\cancel{\cos x}} = 2$ //	j) Challenge: $\sec x\sqrt{\frac{1-\sin^2 y\cdot\sin^2 x}{1+\cos^2 y\cdot\tan^2 x}}$

3. Prove each of the following identities algebraically:

a) $(\sec\theta-\tan\theta)(\sin\theta+1)=\cos\theta$

$\left(\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}\right)(\sin\theta+1)$

$\frac{(1-\sin\theta)(\sin\theta+1)}{\cos\theta}$

$=\frac{\cancel{\sin\theta}+1-\sin^2\theta-\cancel{\sin\theta}}{\cos\theta}$

$=\frac{1-\sin^2\theta}{\cos\theta}$

$=\frac{\cos^2\theta}{\cos\theta}$

$=\cos\theta \quad = RHS.$

b) $\tan x+\cot x=\sec x(\csc x)$

$\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}$

$\frac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}$

$=\frac{1}{\sin\theta\times\cos\theta}$

$=\frac{1}{\cos\theta}\times\frac{1}{\sin\theta}=RHS$

c) $\frac{1}{2}\cot x=\frac{\cos 2x+\sin^2 x}{\sin 2x}$

$\frac{\cos 2x+\sin^2 x}{\sin 2x}$

$=\frac{1-2\sin^2 x+\sin^2 x}{2\sin x\cos x}$

$=\frac{1-\sin^2 x}{2\sin x\cos x}$

$=\frac{\cos x\cdot\cancel{\cos x}}{2\sin x\,\cancel{\cos x}}$

$=\frac{1}{2}\cot x.$

d) $\cot x=\frac{\cos x+\cot x}{1+\sin x}$

$\frac{\cos x+\frac{\cos x}{\sin x}}{1+\sin x}$

$\frac{\left(\frac{\sin x\cos x+\cos x}{\sin x}\right)}{1+\sin x}$

$=\frac{\left(\frac{\cos x(\cancel{\sin x+1})}{\sin x}\right)}{(\cancel{1+\sin x})}$

$=\frac{\cos x}{\sin x}$

$=\cot x.$

e) $\frac{\sec^2 x}{1+\sin x}=\frac{\sec^2 x-\sec x\tan x}{\cos^2 x}$

$\frac{\sec^2 x-\sec x\left(\frac{\sin x}{\cos x}\right)}{\cos^2 x}$

$\frac{\frac{1}{\cos^2 x}-\frac{\sin x}{\cos^2 x}}{1-\sin^2 x.}$

$=\frac{\frac{1}{\cos^2 x}(\cancel{1-\sin x})}{(1+\sin x)(\cancel{1-\sin x})}$

$=\frac{\sec^2 x}{1+\sin x}$ //

f) $\frac{1+\sec x}{\sec x-1}=\frac{1+\cos x}{1-\cos x}$

$\frac{1+\frac{1}{\cos x}}{\frac{1}{\cos x}-1}$

$\frac{\left(\frac{\cos x}{\cos x}+\frac{1}{\cos x}\right)}{\left(\frac{1}{\cos x}-\frac{\cos x}{\cos x}\right)}$

$\frac{\left(\cancel{\frac{1}{\cos x}}\right)(\cos x+1)}{\left(\cancel{\frac{1}{\cos x}}\right)(1-\cos x)}$

$=\frac{\cos x+1}{1-\cos x.}$ //

4. Prove each of the following identities algebraically:

a) $1+\sin 2x=(\sin x+\cos x)^2$

$(\sin x+\cos x)(\sin x+\cos x)$

$=\sin^2 x+\cos^2 x+2\sin x\cos x$

$=1+2\sin x\cos x$

$=1+\sin 2x$

b) $\sin 2x=2\cot x\left(\sin^2 x\right)$

$=2\cot x(\sin^2 x)$

$=2\frac{\cos x}{\cancel{\sin x}}\cdot\sin x\cdot\cancel{\sin x}$

$=2\cos x\sin x$

$=\sin 2x.$

c) $\cos 2x=\frac{1-\tan^2 x}{1+\tan^2 x}$

$\frac{1-\frac{\sin^2 x}{\cos^2 x}}{1+\frac{\sin^2 x}{\cos^2 x}}$

$=\frac{(\cos^2 x-\sin^2 x)\cancel{\left(\frac{1}{\cos^2 x}\right)}}{(\cos^2 x+\sin^2 x)\cancel{\left(\frac{1}{\cos^2 x}\right)}}$

$=\frac{\cos^2 x-\sin^2 x}{(1)}$

$=\cos 2x$ //

d) $\sec^2 x=\frac{2}{1+\cos 2x}$

$=\frac{2}{1+\cos 2x}$

$=\frac{2}{\cancel{1}+(2\cos^2 x-\cancel{1})}$

$=\frac{\cancel{2}}{\cancel{2}\cos^2 x}$

$=\frac{1}{\cos^2 x}$

$=\sec^2 x$ //

e) $\csc x(1+\sin x)=1+\csc x$

$\frac{1}{\sin x}(1+\sin x)$

$=\frac{1}{\sin x}+\frac{\sin x}{\sin x}$

$=\csc x+1=RHS$

f) $\frac{1-\tan x}{1-\cot x}=-\tan x$

$\frac{1-\frac{\sin x}{\cos x}}{1-\frac{\cos x}{\sin x}}$

$=\frac{(\cancel{\cos x-\sin x})\left(\frac{1}{\cos x}\right)}{(\cancel{\sin x-\cos x})\left(\frac{1}{\sin x}\right)}$

$=(-1)\left(\frac{1}{\cos x}\right)\div\left(\frac{1}{\sin x}\right)$

$=(-1)\left(\frac{\sin x}{\cos x}\right)$

$=-\tan x$ //

Note $\frac{a-b}{b-a}=-1$ //

g) $\sin x \tan x + \sec x = \dfrac{\sin^2 x + 1}{\cos x}$	h) $\sin^2 x \cot^2 x = 1 - \sin^2 x$
$\sin x \tan x + \sec x$ $\sin x \left(\frac{\sin x}{\cos x}\right) + \frac{1}{\cos x}$ $= \frac{\sin^2 x}{\cos x} + \frac{1}{\cos x}$ $= \frac{\sin^2 x + 1}{\cos x}$	$\sin^2 x \left(\frac{\cos^2 x}{\sin^2 x}\right) =$ $= \cos^2 x$ $= 1 - \sin^2 x$

5. Prove the following identity: $\dfrac{\cos x}{1+\sin x} + \dfrac{\cos x}{1-\sin x} = 2\sec x$

$\frac{\cos x}{1+\sin x} + \frac{\cos x}{1-\sin x}$

$\frac{\cos x(1-\sin x) + \cos x(1+\sin x)}{(1+\sin x)(1-\sin x)}$

$\frac{\cos x - \cancel{\sin x \cos x} + \cos x + \cancel{\sin x \cos x}}{1-\sin^2 x}$

$= \frac{2\cancel{\cos x}}{\cancel{\cos x}\cdot \cos x}$

$= \frac{2}{\cos x}$

$= 2\sec x$

6. Simplify the following: $\cos\left(x+\frac{\pi}{2}\right) - \cos\left(x-\frac{\pi}{2}\right)$

$\cancel{\cos x \cos\frac{\pi}{2}} - \sin x \sin\frac{\pi}{2} - \left(\cancel{\cos x \cos\frac{\pi}{2}} + \sin x \sin\frac{\pi}{2}\right)$

$= -\sin x(1) - \sin x(1)$

$= -2\sin x$

a) 0 b) $-2\sin x$ c) $2\sin x$ d) $2\cos x$ e) $-2\cos x$

7. Suppose $\cos x = 0$ and $\cos(x+z) = 0.5$. What is the smallest possible positive value of "z"?

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{2}$ (D) $\frac{5\pi}{6}$ (E) $\frac{7\pi}{6}$

$\cos x = 0$

$x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

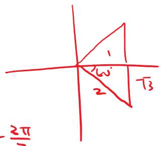
$x = 90^\circ$ or 270°

$\cos(x+z) = 0.5$

$x+z = \frac{\pi}{3}$ or $\frac{5\pi}{3}$

so $\frac{3\pi}{2} + z = \frac{5\pi}{3}$

$z = \frac{5\pi}{3} - \frac{3\pi}{2}$



$\boxed{z = \pi/6}$

8. Suppose that $\sin a + \sin b = \sqrt{5/3}$ and $\cos a + \cos b = 1$. What is $\cos(a-b)$

(A) $\sqrt{\frac{5}{3}} - 1$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{2}{3}$ (E) 1

$(\sin a + \sin b)^2 = \frac{5}{3}$ $(\cos a + \cos b)^2 = 1$

$\sin^2 a + \sin^2 b + 2\sin a \sin b = \frac{5}{3}$ $\cos^2 a + \cos^2 b + 2\cos a \cos b = 1$

$\sin^2 a + \cos^2 a + \sin^2 b + \cos^2 b + 2\cos a \cos b + 2\sin a \sin b = \frac{5}{3} + 1$

$2 + 2(\cos a \cos b + \sin a \sin b) = \frac{8}{3}$

$2(\cos a \cos b + \sin a \sin b) = \frac{8}{3} - \frac{6}{3}$

$2(\cos(a-b)) = \frac{2}{3}$

$\cos(a-b) = \frac{1}{3}$

$\sin^2 a + \cos^2 a + \sin^2 b + \cos^2 b + 2\cos a \cos b + 2\sin a \sin b = \frac{5}{3} + 1$ $\qquad 2(\cos(a-b)) = \frac{2}{3}$

$2 + 2(\cos a \cos b + \sin a \sin b) = \frac{8}{3}$ $\qquad \cos(a-b) = \frac{1}{3}$ //

9. If $\sin x + \cos x = a$, then what is $\sin^3 x + \cos^3 x$ in terms of "a"?

① $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$\sin^3 x + \cos^3 x = (\sin x + \cos x)(\sin^2 x - \sin x \cos x + \cos^2 x)$

$= (a)(\sin^2 x + \cos^2 x - \sin x \cos x)$

$= (a)\left(1 - \frac{a^2}{2} + \frac{1}{2}\right)$

$= (a)\left(\frac{3 - a^2}{2}\right) = \frac{3a - a^3}{2}$ //

② $\sin x + \cos x = a$

$(\sin x + \cos x)^2 = a^2$

$\sin^2 x + \cos^2 x + 2\sin x \cos x = a^2$

$\sin x \cos x = \frac{a^2 - 1}{2} = \frac{a^2}{2} - \frac{1}{2}$

10. Solve for "x" with $0 < x < 2\pi$: $\sin 2x + \cos x = 0$

11. Solve for θ with $0 < \theta < 2\pi$: $2 = 1 + \sin\theta + \sin^2 x + \sin^3 x + \ldots\ldots$

12. Challenge: What is the ordered pair of positive integers (a,b) for which $\frac{a}{b}$ is a reduced fraction and $x = \frac{a\pi}{b}$ is the least positive solution of the equation: $(2\cos 8x - 1)(2\cos 4x - 1)(2\cos 2x - 1)(2\cos x - 1) = 1$?